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HELMHOLTZ ON THE AXIOMS OF GEOMETRY

THE *Academy* journal of the 12th of February, 1870 (vol. i. p. 128), contained a paper by Prof. Helmholtz upon the Axioms of Geometry in a philosophical point of view. The opinions set forth by him were based upon the latest speculations of German geometers, so that a new light seemed to be thrown upon a subject which has long been a cause of ceaseless controversy. While one party of philosophers, especially Kant and the great German school, have pointed to the certainty of geometrical axioms as a proof that these truths must be derived from the conditions of the thinking mind, another party hold that they are empirical, and derived, like other laws of nature, from observation and induction. Helmholtz comes to the aid of the latter party by showing that our Axioms of Geometry will not always be necessarily true; that perhaps they are not exactly true even in this world, and that in other conceivable worlds they would be entirely superseded by a new set of geometrical conditions.

There is no truth, for instance, more characteristic of our geometry than that between two points there can be only one shortest line. But we may imagine the existence of creatures whose bodies should have no thickness, and who should live in the mere superficies of an empty globe. Their geometry would apparently differ from ours; the axiom in question would be found in some cases to fail, because between two points of a sphere diametrically opposite, an infinite number of shortest lines can be drawn. With us, again, the three angles of a rectilineal triangle are exactly equal to two right angles. With them the angles of a triangle would always, more or less, exceed two right angles. In other imaginary worlds the geometrical conditions of existence might be still more strange. We can carry an object from place to place without necessarily observing any change in its shape, but in a spheroidal universe nothing could be carried about without undergoing a gradual distortion, one result of which would be that no two adjoining objects could have a similar form. Creatures living in a pseudo-spherical world would find all our notions about parallel lines incorrect, if indeed they could form a notion of what parallelism means. Nor is Helmholtz contented with sketching what might happen in purely imaginary circumstances. He seems to accept Reimann's startling speculation that perhaps things are not as square and right in this world as we suppose. What should we say if in drawing straight lines to the most distant fixed stars (by means not easy to describe), we found that they would not go exactly straight, so that two lines, when fitted together like rulers, would never coincide, and lines apparently parallel would ultimately intersect? Should we not say that Euclid's axioms cannot hold true? It may be that our space has a certain twist in a fourth dimension unknown to us, which is inappreciable within the bounds of the planetary system, but becomes apparent in stellar distances.

Though Helmholtz gives most of these speculations as due to other writers, he seems, so far as I can gather from his words, to stamp them with the authority of his own high name. It requires a little courage, therefore, to maintain that all these geometrical exercises have no bearing whatever upon the philosophical questions in dispute. Euclid's elements would be neither more nor less true in one such world than another; they would be only more or less applicable. Even in a world where the figures of plane geometry could not exist, the principles of plane geometry might have been developed by intellects such as some men have possessed. And if, in the course of time, the curvature of our space should be detected, it will not falsify our geometry, but merely necessitate the extension of our books upon the subject.

Helmholtz himself gives the clue to the failure in his reasoning. He says: "It is evident that the beings on the spherical surface would not be able to form the notion of geometrical similarity, because they would not know geometrical figures of the same form but different magnitude, except such as were of infinitely small dimensions.' But the exception here suggested is a fatal one. Let us put this question: "Could the dwellers on a spherical world appreciate the truth of the 32nd proposition of Euclid's first book?" I feel sure that, if in possession of human powers of intellect, they could. In large triangles that proposition would altogether fail to be verified, but they could hardly help perceiving that, as smaller and smaller triangles were examined, the spherical excess of the angles decreased, so that the nature of a rectilineal triangle would present itself to them under the form of a limit. The whole of plane geometry would be as true to them as to us, except that it would only be exactly true of infinitely small figures. The principles of the subject would certainly be no more difficult than those of the Differential Calculus, so that if a Euclid could not, at least an Archimedes, a Newton, or a Leibnitz of the spherical world would certainly have composed the books of Euclid, much as we have them. Nay, provided that their figures were drawn sufficiently small, they could verify all truths concerning straight lines just as closely as we can.

I will go a step further, and assert that we are in exactly the same difficulty as the inhabitants of a spherical world. There is not one of the propositions of Euclid which we can verify empirically in this universe. The most perfect mathematical instruments are not two moments of the same form. We are practically unacquainted with straight lines or rectilineal motions or uniform forces. The whole science of mechanics rests upon the notion of a uniform force, but where can we find such a force in operation? Gravity, doubtless, presents the nearest approximation to it; but if we let a body fall through a single foot, we know that the force varies even in that small space, and a strictly correct notion of a uniform force is only got by receding to infinitesimals. I do not think that the geometers of the spherical world would be under any greater difficulties than our mathematicians are in developing a science of mechanics, which is generally true only of infinitesimals. Similarly in all the other supposed universes plane geometry would be approximately true in fact, and exactly true in theory, which is all we can say of this universe. Where parallel lines could not exist of finite magnitude, they would be conceived as of infinitesimal magnitude, and the conception is no more abstruse than that of the direction of a continuous curve, which is never the same for any finite distance. The spheroidal creatures would find the distortion of their own bodies rapidly vanishing as

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the distance of the motion is less, which only amounts to the truth, that a small portion of an ellipse is ultimately undistinguishable from a circle. The truth of the Axioms of Geometry never really comes into question at all, and Helmholtz has merely pointed out circumstances in which the figures treated in plain geometry could not always be practically drawn.

It is a second question whether the dwellers in a spherical world could acquire a notion of three dimensions of space. We must remember that such beings could bear no analogy to us, who have solid bones and flesh, and live upon a solid globe, into which we can penetrate a considerable distance. These beings have no thickness at all, and live in a surface infinitely thinner than the film of a soap bubble, in fact, not thin or thick at all, but devoid of all pretensions to thickness.

There would be nothing at first sight to suggest the threefold dimensions of space, and yet I believe that they could ultimately develop all the truths of solid geometry. They could not fail to be struck with the fact that their geometry of finite figures differed from that of infinitesimals, and an analysis of this mysterious difference would certainly lead them to all the properties of tridimensional space. Indeed, if Riemann, prior to all experience, is able to point out the exact mode in which a curvature of our space would present itself to us, and can furnish us with analytical formulæ upon the subject, why might not the Riemann of the spherical world perform a similar service, and show how the existence of a third dimension was to be detected? It might well be that the inhabitants of the sphere had in the infancy of science never suspected the curvature of the world, and, like our ancestors, had considered the world to be a great plain. In the absence of any experience to that effect, it is certain that the notion of thickness could not be framed any more than we can imagine what a fourth dimension of our space would be like. We have some idea what a world of one dimension would be, because as regards *time* we are in a world of that kind. The characteristic of time is that all intervals beginning and ending at the same moments are equal. But suppose that some people discovered a mysterious way of living which enabled them to live a longer time between the same moments than other people; this could only be accounted for by supposing that they had diverged from the ordinary course of time, like travellers taking a roundabout road. Though in one sense such an occurrence is utterly inconceivable, yet in another sense we can probably anticipate the character of the phenomenon, and the 47th proposition of Euclid's first book would doubtless give the most important truth concerning times thus differing in direction.

With all due deference to so eminent a man as Helmholtz, I must hold that his article includes an *ignoratio elenchi.* He has pointed out the very interesting fact that we can conceive worlds where the Axioms of our Geometry would not apply, and he appears to confuse this conclusion with the falsity of the axioms. Wherever lines are parallel the axiom concerning parallel lines will be true, but if there be no parallel lines in existence, there is nothing of which the truth or falsity of the axiom can come in question. I will not attempt to say by what process of mind we reach the certain truths of geometry, but I am convinced that all attempts to attribute geometrical truth to experience and induction, in the ordinary sense of those words, are transparent failures. Mr. Mill is another philosopher whose views led him to make a bold attempt of the kind. But for real experience and induction he soon substituted an extraordinary process of *mental experimentation*, a handling of ideas instead of things, against which he had inveighed in other parts of his "System of Logic." And the careful reader of Mr. Mill's chapter on the subject (Book II. chapter 5) will find that it involves at the same time the assertion and the denial of the existence of perfectly straight lines. Whatever other doctrines may be true, this doctrine of the purely empirical origin of geometrical truth is certainly false.

W. STANLEY JEVONS

LEIGHTON'S LICHEN-FLORA OF GREAT BRITAIN

The Lichen-Flora of Great Britain, Ireland, and the Channel Islands. By the Rev. W. A. Leighton, F.L.S. (Published for the Author. Shrewsbury, 1871.)

IT falls so rarely to the botanical reviewer in this country to notice works on Lichenology, that we gladly avail ourselves of the present opportunity of introducing to our readers a little unpretentious volume which has the excellent object primarily—"of elevating the knowledge of our insular lichens to a level with that of other branches of our country's flora," and which, moreover, completely vindicates the title of Britain's lichens to at least equal study with the other families of her cryptogamia. Since the publication of Mudd's excellent "Manual" in 1861, the additions made to the lichen-flora of Great Britain and Ireland have been both so numerous and important, that lichenological students have felt the want of some systematic work containing a complete list of the British lichens up to the present date, along with specific diagnoses and other aids to their identification. It was generally felt, moreover, that no fitter authority could undertake so intricate a labour than Mr. Leighton, whose name is identified with lichenological progress in this country by the publication of many important papers of a monographic character, and who is justly regarded, both by home and foreign botanists, as the representative and father of lichenology and lichenologists in Britain. The present work, which we are glad to find is to be followed, in due time, by another which is even more urgently required—a Conspectus of all known lichens throughout the world—is a convenient 12mo volume of about 470 pages, which confines itself mainly to a systematic enumeration, with specific diagnoses, of all the lichens at present known to occur in "Great Britain, Ireland, and the Channel Islands." The nomenclature and classification followed are those of Dr. Nylander, of Paris, who is described as "the *facile princeps* of modern microscopic lichenologists." Succeeding the specific diagnoses, the author cites the leading synonyms; gives references to published plates and fasciculi of dried specimens; narrates the general geographical distribution of species throughout the world, on the one hand, and throughout the three kingdoms on the other; specifies the particular localities of growth in each of these latter kingdoms; and gives, so far as possible, the date of original discovery in Britain, with the name of the discoverer.

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